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Revised geometric measure of entanglement

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Abstract
We present a revised geometric measure of entanglement (RGME) which is just certain generalization of geometric measure of entanglement (GME). The revised version is an entanglement monotone. Some useful inequalities about RGME are deduced. For example, we give the formulae of RGME for the two-parameter class of states in a $2 \otimes n$ quantum system, the two particles' high dimensional maximally entangled mixed state, the isotropic state including the n-particle d-level case and two multipartite bound entangled states. The results show there is a relation $\widetilde{E}_{\sin^2} \leqslant E_{\mathrm{re}}$, and then RGME is an appropriate measure of entanglement.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entanglement, first noted by Einstein–Podolsky–Rosen (EPR) [1] and Schrödinger [2], is an essential feature of quantum mechanics. To date, the entangled states have become a very useful and important resource in quantum computation and quantum communication. As a result, the task of quantifying entanglement has emerged as one conspicuous theme in quantum information theory (QIT). As far as our knowledge, the quantification of entanglement is well understood for bipartite pure states; in a more complex scenario (multipartite systems or mixed states), a complete theory on the characterization and quantification of entanglement presents even great challenge.

Broadly speaking, there are two main approaches taken to the definition of entanglement measures. First is an operational approach [3], in which the measures of entanglement are related to physical tasks that one can perform with a quantum state, as quantum communication, and their representations are the entanglement of formation (EOF) [3–7], the entanglement cost [8, 9], the distillable entanglement [10] and the singlet fraction [11, 12]. The other is an axiomatic approach [13–15], which starts from desirable axioms that a 'good' entanglement measure should satisfy, and then attempts to construct such measures, for example, the RE [13, 14], the negativity [16, 17] and the robustness of entanglement [15, 18] belong to axiomatic measures. Recently, the geometric measure of entanglement (GME) based on the geometry of Hilbert has been proposed [19–23]. Actually, the merit of GME is that it is suitable for any-partite systems with any dimension, although determining it analytically for generic state remains a challenge. However, the closest separable state of GME has a concrete pure state form, while we know most of closest separable states are mixed state, thus it needs somewhat generalization. It deserves addressing that GME has no gentle passage from pure state form to mixed state form. Thus, it is necessary to seek the unification of pure state and mixed state forms and arrive at gratifying requirement. Here, we present an attempt to face this challenge by developing GME.

We are always interested in the investigations of entanglement measure of high dimensional and/or multipartite systems. One of us had ever tried to suggest a generalization of the EOF [24] and a modification of the RE [25]. In this paper, we will propose one way to generalize the GME and make it admit this generalization, that is, the so-called revised GME (RGME). Through the RGME, we quantify the entanglement of two-parameter class of states in $2 \otimes n$ quantum system, two particles' high dimensional maximally entangled mixed state, isotropic state including n-particle d-level case and two multipartite bound entangled states. Furthermore, we obtain an important bound relation for these states. Our results indeed demonstrate that the RGME is an appropriate measure of entanglement.

The paper is organized as follows: in section 2, we review the GME. In section 3, we introduce one generalization called RGME and investigate its properties in detail. Then, we calculate GME, RGME and other entanglement measures through some special classes of quantum states in section 4. In section 5, we summarize some concluding remarks.

2. Geometric measure of entanglement

Exploring a geometric approach to quantify measure of entanglement is first introduced by Shimony [26] in the setting of bipartite pure states, and then generalized to the multipartite setting (via projection operations of various ranks) by Barnum and Linden [27]. Wei and Goldbart further provide the GME on the basis of their works [19–23]. The GME for pure state $|\psi\rangle$ is defined as

$$E_{\sin^2} = 1 - \Lambda^2_{\max} = 1 - \max_{\mathrm{separable}\{\phi\}} \|\langle\phi|\psi\rangle\|^2, \tag{1}$$

where $|\phi\rangle$ is a general n-partite pure state with the form (expanded in the local bases $|e^{(i)}_{p_i}\rangle$)

$$|\phi\rangle = \otimes^n_{i=1}|\phi^{(i)}\rangle = \sum_{p_1 p_2 \cdots p_n} \chi_{p_1 p_2 \cdots p_n} \left|e^{(1)}_{p_1} e^{(2)}_{p_2} \cdots e^{(n)}_{p_n}\right\rangle. \tag{2}$$

In basis independent form, we have

$$\left\langle\psi\left(\otimes^n_{j(\neq i)}\right)\middle|\phi^{(j)}\right\rangle = \Lambda\langle\phi^{(i)}|, \qquad \left(\otimes^n_{j(\neq i)}\left\langle\phi^{(j)}\right)\middle|\psi\right\rangle = \Lambda|\phi^{(i)}\rangle, \tag{3}$$

which are independent of the choice of the local basis. The physical meaning of the GME can be seen from entanglement eigenvalue $\Lambda_{\max}$ which is the cosine of angle between the pure state and its closest separable state. Of course, the stronger the entanglement of state becomes, the farer its closest separable state will be, and the larger will be the angle between them. We remark that determining the entanglement of $|\psi\rangle$ is equivalent to finding the Hartree approximation to the ground state of the auxiliary Hamiltonian $H = -|\psi\rangle\langle\psi|$ [20]. The extension to mixed state can be made via the use of the convex roof (or hull) construction as done for EOF. The essence of the problem is a minimization over all decompositions into pure states, i.e.,

$$\begin{aligned}\rho &= \sum_i p_i|\psi_i\rangle\langle\psi_i|,\\ E(\rho) &= (\mathrm{co}E_{\mathrm{pure}})(\rho) = \min_{\{p_i,\psi_i\}}\sum_i p_i E_{\mathrm{pure}}(|\psi_i\rangle).\end{aligned} \tag{4}$$

So, for the general mixed state, it is difficult to write the clear analytical expression of the GME. It is worth indicating that arbitrary two-qubit mixed state, its GME, has been given [23] by

$$E_{\sin^2} = \tfrac{1}{2}\left(1-\sqrt{1-C(\rho)^2}\right), \tag{5}$$

where $C(\rho)$ is concurrence of an arbitrary two-qubit mixed state defined as follows:

$$C(\rho) = \max\{0, \sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4}\} \tag{6}$$

and $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ are the square roots of eigenvalues of the product $\rho_{AB}\tilde{\rho}_{AB}$,

$$\tilde{\rho}_{AB} = (\sigma_y\otimes\sigma_y)\rho^*_{AB}(\sigma_y\otimes\sigma_y). \tag{7}$$

For the sake of clarifying, we use the GME to calculate some simple two-qubit states.

Example 1.

$$\rho_1 = \lambda|\Phi^\dagger\rangle\langle\Phi^\dagger| + (1-\lambda)|01\rangle\langle 01|, \tag{8}$$

where $|\Phi^\dagger\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$,

$$E_{\sin^2} = \tfrac{1}{2}\left(1-\sqrt{1-C(\rho)^2}\right) = \tfrac{1}{2}\left(1-\sqrt{1-|\lambda|^2}\right). \tag{9}$$

Example 2.

$$\rho_2 = A|01\rangle\langle 01| + (1-A)|10\rangle\langle 10| + \frac{G}{2}(|01\rangle\langle 10| + |10\rangle\langle 01|), \tag{10}$$

where G satisfies $G \leqslant 2\sqrt{A(1-A)}$. This condition ensures the state ρ_2 is semi-definite. Negativity and concurrence of this class of states are equal, i.e. $N(\rho_2) = C(\rho_2) = G$. It is easy to calculate the GME,

$$E_{\sin^2} = \frac{1-\sqrt{1-G^2}}{2}. \tag{11}$$

Based on the requirement of calculating GME of pure state, its closest separable pure state $|\phi\rangle$ is given, i.e. equation (2). However, generally speaking, even in the case of pure states, most of its closest separable states are mixed states, this standpoint has been presented in many references [13, 14]. Only some special states, their closest separable states can be pure states, such as examples in [13, 20, 22]. Thus, it is necessary to break the limitation and generalize the original definition and give satisfaction sufficient to meet needed demand or requirement. In addition, as far as a mixed state is concerned, the convex structure is complicated to compute because it adds the amount of calculation and the magnitude of difficulty. Above definition, equation (4), is a common method to deal with scenario of mixed state. Whether there is a different method which can solve this problem to make it compute easily, at the same time, has gentle passage from pure state form to mixed state form. The answer is affirmative. The main purpose of this paper is to propose certain generalization of GME to avoid facing this embarrassment and make it satisfactory.

3. Revised geometric measure of entanglement

The motivation for constructing the GME is to address the degree of entanglement from a geometric viewpoint, regardless of the number of parties. Yet, there is some room to generalize in the original definition of GME. In this section, we propose the revised GME (RGME) which is just a generalization to make it perfect and we elucidate the revision by some concrete examples.

3.1. Definition

We begin with the revision of GME, which is defined as

$$\widetilde{E}_{\sin^2}(\rho) = \min_{\sigma\in S}(1 - F^2(\rho,\sigma)), \tag{12}$$

where

$$F(\rho,\sigma) = \operatorname{tr}\sqrt{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}, \tag{13}$$

where S denotes the set of separable states. Comparing equation (12) with equation (1), we obtain the relation

$$\Lambda_{\max}^2 = \max_{\sigma\in S} F^2(\rho,\sigma). \tag{14}$$

Remark when density matrices ρ, σ represent pure states, fidelity equals to overlap, above formula can reduce to the definition of the GME, i.e., equation (1). The maximum of overlap is totally different from the maximal fidelity in the sense that the latter's variational range is wider than the form's case, thus our revised GME is more appropriate. The essence of proposed RGME is to calculate the fidelity between given state and its closest separable state. Finally, it reduces to the search of the closest separable state.

Because of the relation between Bures metric and fidelity, and the fact that Bures metric is positive, we have

$$\mathrm{Bures}(\rho) = \sqrt{1 - F^2(\rho,\sigma)}, \tag{15}$$

then the RGME can be expressed as

$$\widetilde{E}_{\sin^2}(\rho) = \min_{\sigma\in S}(\mathrm{Bures}^2(\rho)). \tag{16}$$

Let us see whether the RGME $\widetilde{E}_{\sin^2}$ is a good entanglement measure or not? We know a good entanglement measure should satisfy some properties [28, 29]. It is easy to prove that the RGME indeed satisfies these requirements.

Now, we verify that it is non-increasing under local operation and classical communication (LOCC) transformation using the Uhlmann' theorem [30]. Proof: assume ε is a trace-preserving quantum operation, ρ, σ are density operators. Let $|\psi\rangle, |\varphi\rangle$ be purifications of ρ, σ in a joint system RQ such that $F(\rho,\sigma) = \langle\psi|\varphi\rangle$. Introduce a model environment E for the quantum operation ε which starts in a pure state $|0\rangle$ and interacts with the quantum system Q via a unitary interaction U. Note $U|\psi\rangle|0\rangle$ is a purification of $\varepsilon(\rho)$ and $U|\varphi\rangle|0\rangle$ is a purification of $\varepsilon(\sigma)$. By Uhlmann's theorem, we have

$$\begin{aligned} &F(\varepsilon(\rho),\varepsilon(\sigma)) \geqslant |\langle\psi|\langle 0|UU^{\dagger}|\varphi\rangle|0\rangle| = |\langle\psi|\varphi\rangle| = F(\rho,\sigma)\\ &1 - F(\varepsilon(\rho),\varepsilon(\sigma)) \leqslant 1 - F(\rho,\sigma)\\ &\tilde{E}_{\sin^2}(\varepsilon(\rho)) \leqslant \tilde{E}_{\sin^2}(\rho). \end{aligned} \tag{17}$$

Thus, the proof finishes.

As for LU invariant, it is determined by the property of fidelity. Fidelity is invariant under local unitary (LU) transformation:

$$F(U\rho U^{\dagger}, U\sigma U^{\dagger}) = \mathrm{tr}\sqrt{(U\rho U^{\dagger})^{\frac{1}{2}} U\sigma U^{\dagger} (U\rho U^{\dagger})^{\frac{1}{2}}} \tag{18}$$

and we have

$$\begin{aligned}
\mathrm{tr}\sqrt{(U\rho U^{\dagger})^{\frac{1}{2}} U\sigma U^{\dagger} (U\rho U^{\dagger})^{\frac{1}{2}}} &= \mathrm{tr}\sqrt{U\sqrt{\rho}U^{\dagger}U\sigma U^{\dagger}U\sqrt{\rho}U^{\dagger}} \\
&= \mathrm{tr}\sqrt{U\sqrt{\rho}\sigma\sqrt{\rho}U^{\dagger}} \\
&= \mathrm{tr}\, U\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}U^{\dagger} \\
&= \mathrm{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \\
&= F(\rho, \sigma).
\end{aligned} \tag{19}$$

Therefore, we show that the RGME is a good entanglement measure.

One of virtues of the geometric approach to entanglement is its straightforward adaptability to arbitrary multipartite state (of finite dimensions). The revision of the GME has similar character. There are four differences deserved emphasizing between the RGME and the GME: (1). The RGME use the fidelity to substitute the overlap, then whatever the given state is, pure or mixed state, in light of the relation between fidelity and overlap, the RGME can always be expressed in fidelity form congruously. (2) The revised form abandons the condition that the closest separable state has the form equation (2), even for the case of pure state, say nothing of the mixed state scenario. (3) The revised version does not need the convex hull to consider the case of mixed state like GME which complicates the task of determining mixed state entanglement, whereas the essence of the problem is attributed to find out the closest separable state. (4) For the case of pure state ρ, there always exists a bound condition $\widetilde{E}_{\sin^2}(\rho) \leqslant E_{\sin^2}(\rho)$.

3.2. Examples

In this subsection, we use the RGME to re-calculate the foregoing two examples, the figures are shown for the convenience of analysis. Therein, the closest separable state of these examples are given and testified by the method given in [14].

Example 1: its closest separable state reads as following:

$$\begin{aligned}
\sigma_1 = {} & \frac{\lambda}{2}\left(1-\frac{\lambda}{2}\right)|00\rangle\langle 00| + \frac{\lambda}{2}\left(1-\frac{\lambda}{2}\right)(|00\rangle\langle 11| + |11\rangle\langle 00|) \\
& + \left(1-\frac{\lambda}{2}\right)^2 |01\rangle\langle 01| + \frac{\lambda^2}{4}|10\rangle\langle 10| + \frac{\lambda}{2}\left(1-\frac{\lambda}{2}\right)|11\rangle\langle 11|,
\end{aligned} \tag{20}$$

the RGME is

$$\widetilde{E}_{\sin^2} = 1 - F_{\max}^2 = 1 - \left[\left(1-\frac{\lambda}{2}\right)\sqrt{1-\lambda} + \lambda\sqrt{1-\frac{\lambda}{2}}\right]^2. \tag{21}$$

The analytical expression of RE is given in [9]:

$$E_{\mathrm{re}}(\rho_1) = (\lambda - 2)\log\left(1-\frac{\lambda}{2}\right) + (1-\lambda)\log(1-\lambda). \tag{22}$$

Note that in the whole paper we reckon that the formula of the RE is $E_{\mathrm{re}} = \min_{\sigma\in S}\mathrm{tr}(\rho\log\rho - \rho\log\sigma)$, where log denotes logarithm whose base is two. Now, we analyse the relation about

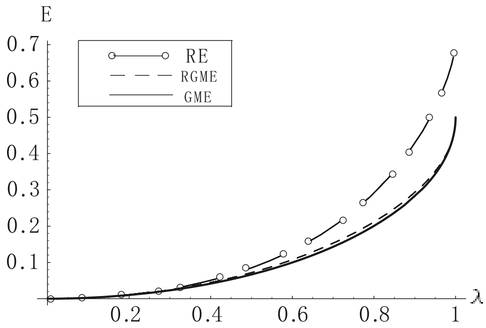

Figure 1. The curves of the GME and RGME almost coincide and lie below the curve of the RE which show that the RGME is better to measure the amount of entanglement of this state.

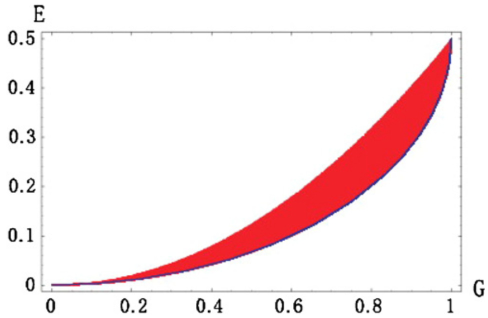

Figure 2. The red field denotes different RGME curves for different random numbers A; the black line represents the GME. The GGME and the GME are coincident when $A \to 1$.

the GME, the RGME and the RE by virtue of figure 1. From the figure, we see that the value of the RE is larger than other two entanglement measures. At the same time, the curve of the GME almost superposes to that of the RGME, which to some extent illustrates that the RGME is reasonable with regard to the GME.

Example 2: its closest separable state is

$$\sigma_2 = A|01\rangle\langle 01| + (1 - A)|10\rangle\langle 10|, \tag{23}$$

the RGME is

$$\widetilde{E}_{\sin^2} = 1 - \left(\sqrt{\frac{1}{2}(1 - 2A + 2A^2 - \sqrt{1 - 4A + 4A^2 + AG^2 - A^2G^2})} + \sqrt{\frac{1}{2}(1 - 2A + 2A^2 + \sqrt{1 - 4A + 4A^2 + AG^2 - A^2G^2})} \right)^2, \tag{24}$$

when choosing A as different random numbers, we get different curves of the RGME shown in figure 2. Obviously, the RGME tends to GME with the increase of A. When $A \to 1$, the RGME superposes to the GME .

In above all examples, the closest separable state of the RGME is identical to that of the RE, and they are mixed states. Yet, we must emphasize that it is not the case in the general

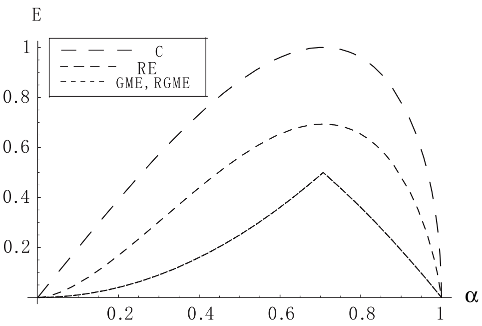

Figure 3. The figures about the RE, the concurrence, the GME and the RGME are shown, in which the curve representing RGME superposes to the curve of GME. Obviously, there is a relation $\widetilde{E}_{\sin^2} = E_{\sin^2} \leqslant E_{\rm re} \leqslant$ concurrence.

situation. Without loss of generality, we are concerned with the state

$$\begin{aligned}\rho &= \alpha^2|00\rangle\langle 00| + \alpha\sqrt{1-\alpha^2}(|00\rangle\langle 11| + |11\rangle\langle 00|) + (1-\alpha^2)|11\rangle\langle 11| \\ &= (\alpha|00\rangle + \sqrt{1-\alpha^2}|11\rangle)(\alpha\langle 00| + \sqrt{1-\alpha^2}\langle 11|) \\ &= |\xi\rangle\langle\xi|,\end{aligned} \tag{25}$$

where $\alpha \in [0, 1]$. Its closest separable state under the RE is

$$\sigma = \alpha^2|00\rangle\langle 00| + (1-\alpha^2)|11\rangle\langle 11|. \tag{26}$$

Naturally, the RE for this state is

$$E_{\rm re} = -\alpha^2 \log\alpha^2 - (1-\alpha^2)\log(1-\alpha^2). \tag{27}$$

However, the closest separable state under the RGME is

$$\sigma' = \left(1 - \sqrt{\frac{1+\sqrt{1-4\alpha^2(1-\alpha^2)}}{2}}\right)|00\rangle\langle 00| + \sqrt{\frac{1+\sqrt{1-4\alpha^2(1-\alpha^2)}}{2}}|11\rangle\langle 11|. \tag{28}$$

By calculation, we know σ' is a disentangled state without reference to α, because all eigenvalues of σ'^{T_B} are non-negative (PPT criterion). Accordingly, the RGME is

$$\widetilde{E}_{\sin^2} = 1 - F_{\max}^2 = \tfrac{1}{2}(1 - \sqrt{1-4\alpha^2(1-\alpha^2)}), \tag{29}$$

which is equal to the GME

$$\begin{aligned}E_{\sin^2} &= \frac{1}{2}(1 - \sqrt{1-4\alpha^2(1-\alpha^2)}) \\ &= \begin{cases} 1-\alpha^2 & \left(\dfrac{\sqrt{2}}{2} < \alpha < 1\right), \\ \alpha^2 & \left(0 \leqslant \alpha \leqslant \dfrac{\sqrt{2}}{2}\right). \end{cases}\end{aligned} \tag{30}$$

If we use the σ' to re-calculate the RE, we get the relation $E_{\rm re}(\rho,\sigma) \leqslant E_{\rm re}(\rho,\sigma')$ which indicates that the closet separable state is indeed the state σ under the RE. In order to demonstrate their relations explicitly, we show figure 3.

3.3. Properties of RGME

It is important to investigate the properties of RGME deeply. In this subsection, we give some propositions with regard to the inequality relations about some measures of entanglement.

Proposition 1. *The RGME and the fidelity satisfy a universal relation for any state ρ, σ:*

$$1 - F(\rho,\sigma) \leqslant \sqrt{\widetilde{E}_{\sin^2}(\rho)}. \tag{31}$$

Proof. Make use of the mathematical formula $\{1 - x \leqslant \sqrt{1-x^2}, 0 \leqslant x \leqslant 1\}$. One can see quickly, for any state ρ, σ,

$$1 - F(\rho,\sigma) \leqslant \sqrt{1 - F^2(\rho,\sigma)}, \tag{32}$$

from which and the definition of the RGME, equation (12), the proposition follows. □

Proposition 2. *The RGME for bipartite pure states is smaller than the entanglement of formation or the relative entropy of entanglement, i.e.,*

$$\widetilde{E}_{\sin^2}(\rho) \leqslant E_f(\rho) = E_{\text{re}}(\rho) \tag{33}$$

and the equality is valid only when ρ is separable state.

Proof. When $\rho = |\psi\rangle\langle\psi|$, and $|\psi\rangle$ is decomposed as A and B parts, we have

$$\text{Bures}(\rho) = \sqrt{1 - F^2(\rho,\sigma)} \leqslant -\tfrac{1}{2}\,\text{tr}(\rho_A \log \rho_A) = \tfrac{1}{2} S_A(\rho), \tag{34}$$

where $S_A(\rho)$ is the von Neumann reduced entropy. In term of equation (32), then we have

$$1 - F(\rho,\sigma) \leqslant \tfrac{1}{2} S_A(\rho) \tag{35}$$

under the condition that ρ is bipartite pure state; accordingly, we can deduce that the RGME satisfies the inequality

$$\widetilde{E}_{\sin^2}(\rho) \leqslant 1 - \left(1 - \frac{S_A(\rho)}{2}\right)^2 = S_A(\rho) - \frac{S_A^2(\rho)}{4} \leqslant S_A(\rho) \tag{36}$$

and the equality is valid only when $S_A(\rho)$ is zero, that is, ρ is separable. □

Due to the existence of Schmidt decomposition [31] in bipartite pure state system, the EF is equal to the von Neumann reduced entropy and the RE, i.e. $S_A(\rho) = E_f(\rho) = E_{\text{re}}(\rho)$, thus, we arrive at the desired relation.

Proposition 3. *The RGME is smaller than the trace distance for any pure state $|\psi\rangle$, i.e.*

$$\widetilde{E}_{\sin^2}(|\psi\rangle) \leqslant D(|\psi\rangle,\sigma). \tag{37}$$

Proof. In the case of pure state $\rho = |\psi\rangle\langle\psi|$, there exits another relation

$$1 - F^2(|\psi\rangle,\sigma) \leqslant \tfrac{1}{2}\,\text{tr}(||\psi\rangle\langle\psi| - \sigma|) = D(|\psi\rangle,\sigma), \tag{38}$$

where $D(|\psi\rangle,\sigma)$ is trace distance between state $|\psi\rangle$ and σ. In light of the definition of the RGME, we obtain the sought inequality finally. □

As we know, among many measures of entanglement, the von Neumann entropy is very important and it has extensive application. Equations (36) and (37) prompt us to ask what is the relation between $D(\rho,\sigma)$ and $S(\rho)$?

Proposition 4. *The von Neumann entropy and distance trace satisfy the inequality*

$$S(\rho) \leqslant 2D(\rho,\sigma) + \frac{1}{e}, \tag{39}$$

under the condition that σ is pure state and the dimension of any state ρ fulfils $d \geqslant 4$.

Proof. Due to *Fannes' inequality*, suppose ρ and σ are density matrices such that the trace distance between them satisfies $\frac{1}{2}\operatorname{tr}(|\rho-\sigma|) \leqslant \frac{1}{e}$, then

$$|S(\rho) - S(\sigma)| \leqslant \tfrac{1}{2}\operatorname{tr}|\rho-\sigma| \cdot \log d + \eta\left(\tfrac{1}{2}\operatorname{tr}|\rho-\sigma|\right), \tag{40}$$

where d is the dimension of the Hilbert space, and $\eta(x) = -x\log x$. Removing the restriction that $\frac{1}{2}\operatorname{tr}(|\rho-\sigma|) \leqslant \frac{1}{e}$, there is an inequality

$$|S(\rho) - S(\sigma)| \leqslant \frac{1}{2}\operatorname{tr}|\rho-\sigma| \cdot \log d + \frac{1}{e}. \tag{41}$$

Using the weaker inequality, when $S(\sigma) = 0$, i.e. σ is a pure state, it yields

$$|S(\rho)| \leqslant \frac{1}{2}\operatorname{tr}|\rho-\sigma| \cdot \log d + \frac{1}{e}. \tag{42}$$

Note that in the definition—and throughout this paper—logarithms indicated by 'log' are taken to base two, while 'ln' indicates a natural logarithm. When $\frac{\log d}{2} \geqslant 1$, we deduce the relation $d \geqslant 4$, then we obtain a relation between the von Neumann entropy and distance trace

$$S(\rho) \leqslant \operatorname{tr}|\rho-\sigma| = 2D(\rho,\sigma) + \frac{1}{e}, \tag{43}$$

hence the proposition is proved. $\square$

As a subsidiary product, we combine above deduction with the facts

$$F(\rho,\sigma) + D(\rho,\sigma) \geqslant 1, \qquad F^2(\rho,\sigma) + D^2(\rho,\sigma) \leqslant 1, \tag{44}$$

then the unambiguous relation under the condition ρ in pure state becomes

$$(1-F)^2 \leqslant 1 - F \leqslant \widetilde{E}_{\sin^2} \leqslant 1 - F^2 \leqslant D \leqslant \sqrt{1-F^2} \leqslant \sqrt{D}. \tag{45}$$

Thus, we have investigated the relations between different measures of entanglement. In view of their different physical meaning for measuring the amount of entanglement, we believe these relations may imply much in many problems such as comparison about different measures of entanglement, discussion about the bound of different measures of entanglement.

4. RGME of some special classes of states

Progress in the quantification of entanglement for a mixed state has resided primarily in the domain of bipartite systems. If we can formulate the universal measures of many-particle system and multipartite system entanglement, they would have many applications [32]. One purpose of this paper is to achieve some analytical form of the RGME for some special cases that are interesting in theory. Concretely, we use mathematical induction method to obtain the expressions of the RGME for two-parameter class of states in $2 \otimes n$ quantum system, bipartite maximally entangled mixed state, isotropic state including n-particle d-level case and two multipartite bound entangled states; the relation between the RGME and the corresponding GME is also obtained. At the same time, we obtain an important conclusion that the RE is an upper bound on the RGME for these states. Based on these results, we can clearly see the advantages of RGME relative to other measures of entanglement.

4.1. RGME of two-parameter class of states in $2 \otimes n$ quantum system

Now, we consider the class of states with two real parameters α and γ in $2 \otimes n$ quantum system. A finite-dimensional truncation of a single two level atom interacting with a single-mode quantized field [33] can be regarded as a $2 \otimes n$ quantum system.

First, we deal with the simple $n = 3$ case. For $2 \otimes 3$ quantum system, two parameters state [34] can be expressed as

$$\tilde{\rho} = \beta|\psi^+\rangle\langle\psi^+| + \gamma|\psi^-\rangle\langle\psi^-| + \beta(|00\rangle\langle 00| + |11\rangle\langle 11|) + \alpha(|02\rangle\langle 02| + |12\rangle\langle 12|), \tag{46}$$

where

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \qquad |\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{47}$$

and β is dependent on α and γ by the unit trace condition,

$$2\alpha + 3\beta + \gamma = 1, \qquad 0 \leqslant \alpha \leqslant \tfrac{1}{2}, \tag{48}$$

remark when $\frac{1}{2} \leqslant \gamma \leqslant 1$ the state is entangled. Similar to the method in [35], we know that the closest separable state of $\tilde{\rho}$ has the following form:

$$\tilde{\sigma}^* = p_1|\psi^+\rangle\langle\psi^+| + p_2|\psi^-\rangle\langle\psi^-| + p_3|00\rangle\langle 00| + p_4|11\rangle\langle 11| + p_5|02\rangle\langle 02| + p_6|12\rangle\langle 12|, \tag{49}$$

where $\sum_i p_i = 1$.

We use convex programming method to determine the concrete form of p_i. The basic idea is: by using the positive definition or semi-positive definition of partial transpose of separable state $\tilde{\sigma}^*$ and Lagrangian multiplier limitation method to seek for the solution of p_i, and then, we find out the closest separable state for this class of state. The separable criterion is a necessary and significant condition for $2 \otimes 2$ and $2 \otimes 3$ quantum systems, so there are much more limitations in the research field. If the value p_i makes $\tilde{\sigma}^* = \sum p_i|\psi_i\rangle\langle\psi_i|$ nonseparable, then this method is invalid.

By calculation, the closest separable state of $\tilde{\rho}$ can be expressed as

$$\begin{aligned}\tilde{\sigma}^* = {} & \alpha(|02\rangle\langle 02| + |12\rangle\langle 12|) + \frac{3\beta + \gamma}{2}|\psi^-\rangle\langle\psi^-| \\ & + \frac{3\beta + \gamma}{6}(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|).\end{aligned} \tag{50}$$

We need to point out that the detail process was given in our classmate' unpublished thesis [36] that is provided in appendix A.

Then, the RE of $\tilde{\rho}$ is

$$\begin{aligned}E_{\mathrm{re}}(\tilde{\rho}) &= S(\tilde{\rho}\|\tilde{\sigma}^*) = \mathrm{tr}(\tilde{\rho}\log\tilde{\rho}) - \mathrm{tr}(\tilde{\rho}\log\tilde{\sigma}^*) \\ &= 3\beta\log\left[\frac{6\beta}{3\beta + \gamma}\right] + \gamma\log\left[\frac{2\gamma}{3\beta + \gamma}\right] \\ &= (1 - 2\alpha)\log\left[\frac{2(1 - \gamma - 2\alpha)}{1 - 2\alpha}\right] + \gamma\log\left[\frac{\gamma}{1 - \gamma - 2\alpha}\right].\end{aligned} \tag{51}$$

Now, let us consider complex case, i.e., two-parameter class of states in $2 \otimes n$ quantum system [34] for $n \geqslant 3$, which can be obtained from an arbitrary state in $2 \otimes n$ quantum system by LOCC and are invariant under all unitary operations with the form $U \otimes U$ on $2 \otimes n$ quantum system:

$$\begin{aligned}\rho = {} & \alpha\sum_{i=0}^{1}\sum_{j=2}^{n-1}|ij\rangle\langle ij| + \beta(|00\rangle\langle 00| + |11\rangle\langle 11|) + \frac{\beta + \gamma}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|) \\ & + \frac{\beta - \gamma}{2}(|01\rangle\langle 10| + \langle 10|\langle 01|),\end{aligned} \tag{52}$$

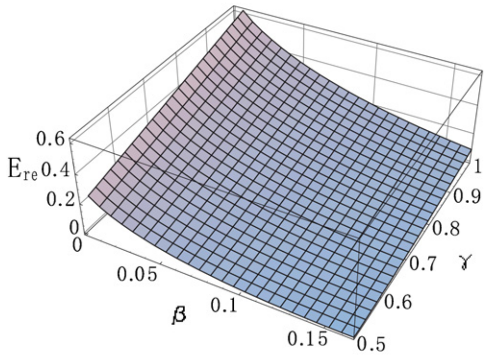

Figure 4. The relative entropy of entanglement (RE) of two-parameter class of states in $2 \otimes n$ quantum system.

where $\{|ij\rangle : i = 0, 1, j = 0, 1, \ldots, n-1\}$ is an orthonormal basis for $2 \otimes n$ quantum system, and the coefficients satisfy the relation

$$2(n-2)\alpha + 3\beta + \gamma = 1. \tag{53}$$

Remark when $\alpha = 0$, this state equals the Werner state in $2 \otimes 2$ quantum system for $0 \leqslant \gamma \leqslant 1$. The state ρ is entangled and distillable if and only if $\frac{1}{2} < \gamma \leqslant 1$. We guess its closest separable state is

$$\sigma^* = \alpha \sum_{i=0}^{1} \sum_{j=2}^{n-1} |ij\rangle\langle ij| + \frac{3\beta + \gamma}{6}(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|) + \frac{3\beta + \gamma}{2}|\psi^-\rangle\langle\psi^-|. \tag{54}$$

The above expression is indeed the closest separable state under the RE for the two-parameter class of states in $2 \otimes n$ quantum system which has been analysed in our previous work [36] (see appendix B).

The RE of two-parameter state is

$$\begin{aligned} E_{\mathrm{re}}(\rho) = S(\rho\|\sigma^*) &= \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma^*) \\ &= (1 - 2(n-2)\alpha) \log \left[\frac{2(1 - \gamma - 2(n-2)\alpha)}{1 - 2(n-2)\alpha}\right] \\ &\quad + \gamma \log \left[\frac{\gamma}{1 - \gamma - 2(n-2)\alpha}\right] \\ &= 3\beta \log \left(\frac{6\beta}{3\beta + \gamma}\right) + \gamma \log \left(\frac{2\gamma}{3\beta + \gamma}\right) \end{aligned} \tag{55}$$

where

$$2(n-2)\alpha + 3\beta + \gamma = 1, \qquad 0 \leqslant \alpha \leqslant 1/(2n-4), \qquad \tfrac{1}{2} \leqslant \gamma \leqslant 1, \tag{56}$$

we draw the three-dimensional picture of RE in figure 4.

Now we begin to calculate the analytical expression of the RGME for two-parameter class of states in $2 \otimes n$ quantum system. Since we obtain the expression of the RE, we want to ask whether the closest separable state under the RE is also the closest separable state under the RGME? The answer is yes for this state. We present a proof in the following context.

From the definition of the RGME, we know the closest separable state under the RGME is that under the fidelity. To prove the closest disentangled state to ρ under the fidelity metric is σ^*, that is equation (54), we consider a slight variation around σ^* of the form $\sigma_\lambda = (1-\lambda)\sigma^* + \lambda\sigma$ where σ is any separable state, then we just need to prove

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathrm{tr}\left\{\sqrt{\sqrt{\rho}\sigma_\lambda\sqrt{\rho}}\right\} \leqslant 0. \tag{57}$$

Proof.

$$\sqrt{\rho}\sigma_\lambda\sqrt{\rho} = \rho^{\frac{1}{2}}[(1-\lambda)\sigma^* + \lambda\sigma]\rho^{\frac{1}{2}} = (1-\lambda)\rho^{\frac{1}{2}}\sigma^*\rho^{\frac{1}{2}} + \lambda\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}. \tag{58}$$

By choosing the appropriate basis sequence, σ^* and ρ can be expressed in the following matrix form:

$$\sigma^* = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad \rho = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \tag{59}$$

where

$$A_1 = \begin{pmatrix} \frac{3\beta+\gamma}{6} & 0 & 0 & 0 \\ 0 & \frac{3\beta+\gamma}{3} & -\frac{3\beta+\gamma}{6} & 0 \\ 0 & -\frac{3\beta+\gamma}{6} & \frac{3\beta+\gamma}{3} & 0 \\ 0 & 0 & 0 & \frac{3\beta+\gamma}{6} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \frac{\beta+\gamma}{2} & \frac{\beta-\gamma}{2} & 0 \\ 0 & \frac{\beta-\gamma}{2} & \frac{\beta+\gamma}{2} & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \tag{60}$$

$$A_2 = \begin{pmatrix} \alpha & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \alpha & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha \end{pmatrix}.$$

In addition, we can write any separable state σ as diagonal matrix block form

$$\sigma = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \tag{61}$$

then we can deduce

$$\begin{aligned} \sqrt{(1-\lambda)\rho^{\frac{1}{2}}\sigma^*\rho^{\frac{1}{2}} + \lambda\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}} &= \sqrt{1-\lambda}\begin{pmatrix} \left(B_1^{\frac{1}{2}}A_1B_1^{\frac{1}{2}}\right)^{\frac{1}{2}} & 0 \\ 0 & \left(B_2^{\frac{1}{2}}A_2B_2^{\frac{1}{2}}\right)^{\frac{1}{2}} \end{pmatrix} \\ &\quad + \sqrt{\lambda}\begin{pmatrix} \left(B_1^{\frac{1}{2}}X_1B_1^{\frac{1}{2}}\right)^{\frac{1}{2}} & 0 \\ 0 & \left(B_2^{\frac{1}{2}}X_2B_2^{\frac{1}{2}}\right)^{\frac{1}{2}} \end{pmatrix}. \end{aligned} \tag{62}$$

We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathrm{tr}\sqrt{\sqrt{\rho}\sigma_\lambda\sqrt{\rho}} &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\sqrt{1-\lambda}\left[F_{A_1} + F_{A_2}\right] + \sqrt{\lambda}\left[F_{X_1} + F_{X_2}\right]\right) \\ &= -\frac{1}{2}(1-\lambda)^{-\frac{1}{2}}\left(F_{A_1} + F_{A_2}\right) + \frac{1}{2}\lambda^{-\frac{1}{2}}\left(F_{X_1} + F_{X_2}\right), \end{aligned} \tag{63}$$

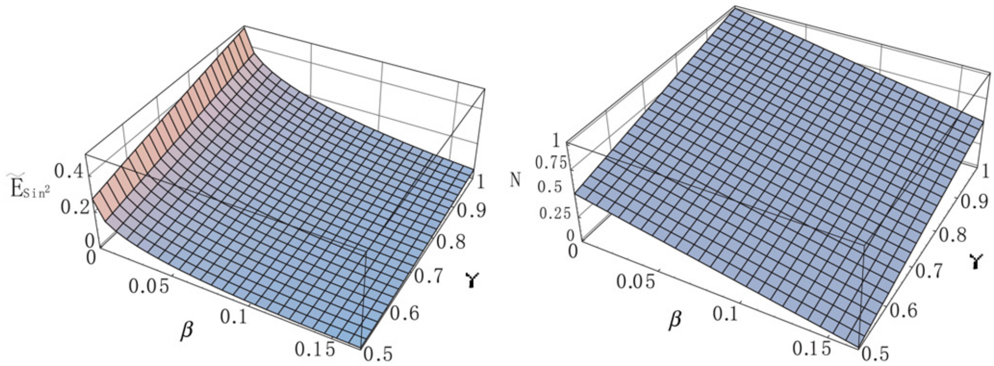

Figure 5. Revised geometric measure of entanglement (RGME) and negativity for the two-parameter class of states in $2 \otimes n$ quantum system, respectively.

where $F_{A_1} = \operatorname{tr}\sqrt{B_1^{\frac{1}{2}} A_1 B_1^{\frac{1}{2}}}$, $F_{A_2} = \operatorname{tr}\sqrt{B_2^{\frac{1}{2}} A_2 B_2^{\frac{1}{2}}}$ and $F_{X_1} = \operatorname{tr}\sqrt{B_1^{\frac{1}{2}} X_1 B_1^{\frac{1}{2}}}$, $F_{X_2} = \operatorname{tr}\sqrt{B_2^{\frac{1}{2}} X_2 B_2^{\frac{1}{2}}}$, respectively. We put $\lambda = 0$ into the above expression, we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \operatorname{tr}\left\{\sqrt{\sqrt{\rho}\sigma_\lambda\sqrt{\rho}}\right\} = -\frac{1}{2}\left(F_{A_1} + F_{A_2}\right). \tag{64}$$

Because the fidelity is always larger than or equal to 0, finally we obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \operatorname{tr}\left\{\sqrt{\sqrt{\rho}\sigma_\lambda\sqrt{\rho}}\right\} \leqslant 0. \tag{65}$$

The proof comes to an end. □

The RGME for the two-parameter class of states in $2 \otimes n$ quantum system is thus

$$\begin{aligned}\widetilde{E}_{\sin^2} &= 1 - F_n^2 = 1 - \left(2(n-2)\alpha + \frac{3\sqrt{\beta(3\beta+\gamma)}}{\sqrt{6}} + \frac{\sqrt{3\beta\gamma+\gamma^2}}{\sqrt{2}}\right)^2 \\ &= 1 - \left(1 - \gamma - 3\beta + \frac{3\sqrt{\beta(3\beta+\gamma)}}{\sqrt{6}} + \frac{\sqrt{\gamma(3\beta+\gamma)}}{\sqrt{2}}\right)^2,\end{aligned} \tag{66}$$

where

$$0 \leqslant \alpha \leqslant \frac{1}{2n-4}, \qquad -\frac{1}{3} \leqslant \beta \leqslant \frac{1}{6}, \qquad \frac{1}{2} \leqslant \gamma \leqslant 1. \tag{67}$$

The concrete proof process is given in appendix C.

In virtue of the results of previous works [34, 36], we know that the negativity of this class of state is

$$\begin{aligned}N &= \left(2(n-2)\alpha + 3\left|\frac{\beta+\gamma}{2}\right| + \frac{1}{2}|3\beta-\gamma|\right) - 1 \\ &= -3\beta - \gamma + 3\left|\frac{\beta+\gamma}{2}\right| + \frac{1}{2}|3\beta-\gamma|.\end{aligned} \tag{68}$$

Of course, the corresponding three-dimensional picture of the RGME and the negativity can be drawn in figure 4.

We can know the relation of three measures of entanglement for the two-parameter class of states in $2 \otimes n$ quantum system by drawing figure 5, i.e. $\widetilde{E}_{\sin^2} \leqslant E_{\mathrm{re}} \leqslant$ negativity.

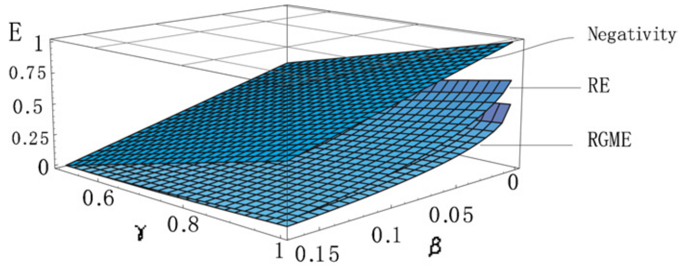

Figure 6. RGME, RE, negativity for the two-parameter class of states in $2 \otimes n$ quantum system. There is a relation $\widetilde{E}_{\sin^2} \leqslant E_{\rm re} \leqslant$ negativity clearly.

The tight upper bound of EOF for the $2 \otimes n$ quantum system is given in [37]: $E_f \leqslant$ negativity. And [38] presents a lower bound for EOF on $2 \otimes n$ system and compares this lower bound with the RE: $E_{\rm re} \leqslant E_f$. Besides, we know that any $2 \otimes n$ quantum state can be transformed to two-parameter class of states, equation (52), by LOCC. Based on the requirement that the measure of entanglement is not increased under LOCC, we acquire the relation for the two-parameter class of states in $2 \otimes n$ quantum system:

$$\widetilde{E}_{\sin^2} \leqslant E_{\rm re} \leqslant E_f \leqslant \text{negativity}. \tag{69}$$

From above analysis and comparison, we see that using the RGME to measure the entanglement is more appropriate for the two-parameter class of states in $2 \otimes n$ quantum system.

4.2. RGME of maximally entangled mixed state

Ishizaka and Hiroshima first introduce the concept of the maximally entangled mixed state [39] for which no more entanglement can be created by global unitary operation, that is, acting on the system as a whole.

In theory, by investigating the maximally entangled mixed state, we can know the bounds on how the degree of mixing of a state limits its entanglement. In practice, the mixture of the density matrix is inevitably increased by the coupling between the quantum system and its surrounding environment in all realistic systems. Therefore, it is extremely important to understand the nature of entanglement for general mixed states between two extremes of pure states and a maximally mixed state.

Here, we extend results of two-parameter class of states in $2 \otimes n$ $(n > 2)$ quantum system to the maximally entangled mixed state [40] of two particles' high dimensional situation. Let the eigenvalue decomposition of ρ, i.e. equation (52), be

$$\rho = \Phi \Lambda \Phi^{\dagger}, \tag{70}$$

where the eigenvalues λ_i are sorted in nonascending order. Obviously, we are capable of obtaining a mixed state ρ' which achieves maximal negativity by applying the following global unitary transformation:

$$U = (U_1 \otimes U_2)\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1/\sqrt{2} & \cdots & 0 & 0 & \cdots & 0 & 1/\sqrt{2} & 0 \\ 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ -1/\sqrt{2} & \cdots & 0 & 0 & \cdots & 0 & 1/\sqrt{2} & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} D_\phi \Phi^\dagger, \tag{71}$$

where U_1, U_2 are the two subsystem local unitary operations, respectively. D_ϕ is a unitary diagonal matrix. The transformed state is

$$\begin{aligned} \rho' = U\rho U^\dagger &= \lambda_4|00\rangle\langle 00| + \frac{\lambda_1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|) \\ &\quad + \lambda_3|0(n-1)\rangle\langle 0(n-1)| + \lambda_2|1(n-1)\rangle\langle 1(n-1)| \\ &= \lambda_4|00\rangle\langle 00| + \lambda_1|\psi^-\rangle\langle\psi^-| + \lambda_3|0(n-1)\rangle\langle 0(n-1)| \\ &\quad + \lambda_2|1(n-1)\rangle\langle 1(n-1)|, \end{aligned} \tag{72}$$

where $\lambda_i (i = 1, 2, 3, 4)$ are four eigenvalues which satisfy

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1. \tag{73}$$

The negativity of ρ' is same as the counterpart of two-qubit situation [40]. Negativity of this state is also the maximum:

$$N(\rho') = \max(0, \sqrt{(\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_4)^2} - \lambda_2 - \lambda_4). \tag{74}$$

The closest separable state of ρ' has the following form:

$$\begin{aligned} \sigma' = {} & p_4|00\rangle\langle 00| + p_5|11\rangle\langle 11| + p_1|\psi^-\rangle\langle\psi^-| \\ & + p_3|0(n-1)\rangle\langle 0(n-1)| + p_2|1(n-1)\rangle\langle 1(n-1)|. \end{aligned} \tag{75}$$

By convex programming method, we get the concrete form of state σ',

$$\begin{aligned} \sigma' = {} & \frac{(\lambda_1 + 2\lambda_4)^2}{4(\lambda_1 + \lambda_4)}|00\rangle\langle 00| + \frac{\lambda_1^2}{4(\lambda_1 + \lambda_4)}|11\rangle\langle 11| \\ & + \frac{\lambda_1(\lambda_1 + 2\lambda_4)}{4(\lambda_1 + \lambda_4)}(|01\rangle\langle 01| + |10\rangle\langle 10| - |01\rangle\langle 10| - |10\rangle\langle 01|) \\ & + \lambda_3|0(n-1)\rangle\langle 0(n-1)| + \lambda_2|1(n-1)\rangle\langle 1(n-1)|, \end{aligned} \tag{76}$$

the RE of ρ' is

$$E_{\mathrm{re}}(\rho') = \lambda_1 \log\frac{2(\lambda_1 + \lambda_4)}{(\lambda_1 + 2\lambda_4)} + \lambda_4 \log\frac{4\lambda_4(\lambda_1 + \lambda_4)}{(\lambda_1 + 2\lambda_4)^2}. \tag{77}$$

It is easy to testify that the closest separable state under the RE is same as that under the fidelity by the same method for the two-parameter class of states in $2 \otimes n$ quantum system.

We can prove that the fidelity of the maximally entangled mixed state ρ' is

$$F = \lambda_2 + \lambda_3 + \frac{(\lambda_1 + 2\lambda_4)}{2}\sqrt{\frac{\lambda_4}{\lambda_1 + \lambda_4}} + \lambda_1\sqrt{\frac{\lambda_1 + 2\lambda_4}{2(\lambda_1 + \lambda_4)}}. \tag{78}$$

Above formula of fidelity is without reference to n (see appendix D). Then, the RGME is

$$\widetilde{E}_{\sin^2} = 1 - \left(1 - \lambda_1 - \lambda_4 + \frac{(\lambda_1 + 2\lambda_4)}{2}\sqrt{\frac{\lambda_4}{\lambda_1 + \lambda_4}} + \lambda_1\sqrt{\frac{\lambda_1 + 2\lambda_4}{2(\lambda_1 + \lambda_4)}}\right)^2, \quad (79)$$

where $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$. By comparing the amount of entanglement for different entanglement measures, equations (77) and (79), we find out that the RE is an upper bound on the RGME for this special state, i.e.,

$$\widetilde{E}_{\sin^2}(\rho) \leqslant E_{\mathrm{re}}(\rho), \quad (80)$$

which also shows that the RGME we use to measure the entanglement is more appropriate for this class of state system.

4.3. RGME of isotropic states

Since the isotropic states are put forward, the properties of isotropic states have been investigated and they have many applications in different fields [44–46]. The state is called isotropic because it is invariant under any $U_A \otimes U_B^*$ transformation:

$$(U_A \otimes U_B^*)\rho_\alpha(U_A \otimes U_B^*)^\dagger = \rho_\alpha, \quad (81)$$

where U is a unitary operator and U^* is its conjugate [45]. In essence, the isotropic states are a class of mixed states which are convex mixtures of the maximally mixed state, $I_{d^2} = (I \otimes I)/d^2$, with a maximally entangled state $|\phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$.

Here, we present some new results about measure of entanglement for the isotropic states including multi-particle and high dimensional generalization. Simultaneously, we review some other measures of entanglement about the isotropic states.

First of all, for all isotropic qubit state

$$\begin{aligned}\rho_\alpha &= \alpha\left|\phi_+^2\right\rangle\!\left\langle\phi_+^2\right| + \frac{1-\alpha}{4}I \\ &= \begin{pmatrix} \frac{1+\alpha}{4} & 0 & 0 & \frac{\alpha}{2} \\ 0 & \frac{1-\alpha}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\alpha}{4} & 0 \\ \frac{\alpha}{2} & 0 & 0 & \frac{1+\alpha}{4} \end{pmatrix},\end{aligned} \quad (82)$$

where $\left|\phi_+^2\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

We know when $-\frac{1}{3} \leqslant \alpha \leqslant \frac{1}{3}$, ρ_α is a separable state; when $\frac{1}{3} < \alpha \leqslant 1$, ρ_α is an entangled state, so the closest separable state is $\sigma = \rho_{\frac{1}{3}}$. The RGME is

$$\widetilde{E}_{\sin^2} = 1 - F^2 = 1 - \left(\frac{1}{2}\sqrt{\frac{3(1-\alpha)}{2}} + \frac{1}{2}\sqrt{\frac{1+3\alpha}{2}}\right)^2. \quad (83)$$

For isotropic qutrit entangled state:

$$\rho_\alpha = \alpha\left|\phi_+^3\right\rangle\!\left\langle\phi_+^3\right| + \frac{1-\alpha}{9}I$$
$$= \begin{pmatrix} \frac{1+2\alpha}{9} & 0 & 0 & 0 & \frac{\alpha}{3} & 0 & 0 & 0 & \frac{\alpha}{3} \\ 0 & \frac{1-\alpha}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\alpha}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\alpha}{9} & 0 & 0 & 0 & 0 & 0 \\ \frac{\alpha}{3} & 0 & 0 & 0 & \frac{1+2\alpha}{9} & 0 & 0 & 0 & \frac{\alpha}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1-\alpha}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-\alpha}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-\alpha}{9} & 0 \\ \frac{\alpha}{3} & 0 & 0 & 0 & \frac{\alpha}{3} & 0 & 0 & 0 & \frac{1+2\alpha}{9} \end{pmatrix}, \tag{84}$$

where $\left|\phi_+^3\right\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$.

We know when $-\frac{1}{8} \leqslant \alpha \leqslant \frac{1}{4}$, ρ_α is a separable state; when $\frac{1}{4} < \alpha \leqslant 1$, ρ_α is an entangled state. The closest separable state is $\sigma = \rho_{\frac{1}{4}}$. The RGME is

$$\widetilde{E}_{\sin^2} = 1 - F^2 = 1 - \left(\frac{1}{3}\sqrt{\frac{16(1-\alpha)}{3}} + \frac{1}{3}\sqrt{\frac{1+8\alpha}{3}}\right)^2. \tag{85}$$

For isotropic qu-quartit entangled state:

$$\rho_\alpha = \alpha\left|\phi_+^4\right\rangle\!\left\langle\phi_+^4\right| + \frac{1-\alpha}{16}I, \tag{86}$$

where $\left|\phi_+^4\right\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$.

We know when $-\frac{1}{15} \leqslant \alpha \leqslant \frac{1}{5}$, ρ_α is a separable state; when $\frac{1}{5} < \alpha \leqslant 1$, ρ_α is an entangled state. The closest separable state is $\sigma = \rho_{\frac{1}{5}}$. The RGME is

$$\widetilde{E}_{\sin^2} = 1 - F^2 = 1 - \left(\frac{1}{4}\sqrt{\frac{45(1-\alpha)}{4}} + \frac{1}{4}\sqrt{\frac{1+15\alpha}{4}}\right)^2. \tag{87}$$

For the $d \times d$ isotropic state:

$$\rho_\alpha = \frac{1}{d^2}\left(1 + \frac{d}{2}\alpha\Gamma\right), \tag{88}$$

where $\Gamma = \sum_{i=1}^{d^2-1} c_i\gamma^i \otimes \gamma^i$, $c_i = \pm 1$, γ^i is Gell–Mann matrix.

We know when $-\frac{1}{d^2+1} \leqslant \alpha \leqslant \frac{1}{d+1}$, ρ_α is a separable state; when $\frac{1}{d+1} < \alpha \leqslant 1$, ρ_α is an entangled state. The closest separable state is

$$\sigma = \rho_{\frac{1}{d+1}} = \frac{1}{d^2}\left(1 + \frac{d}{2\,(d+1)}\Gamma\right). \tag{89}$$

It is easy to get the analytical expression of the fidelity

$$F = \frac{d^2-1}{d}\sqrt{\frac{(1-\alpha)}{d(d+1)}} + \frac{1}{d}\sqrt{\frac{1+(d^2-1)\alpha}{d}}, \tag{90}$$

The RGME is

$$\widetilde{E}_{\sin^2} = 1 - F^2 = 1 - \left(\frac{d^2-1}{d}\sqrt{\frac{(1-\alpha)}{d(d+1)}} + \frac{1}{d}\sqrt{\frac{1+(d^2-1)\alpha}{d}}\right)^2. \tag{91}$$

Furthermore, the isotropic state can be expressed as [17, 31]

$$\rho_{\text{iso}} = \frac{1-F'}{d^2-1}(I - |\phi^+\rangle\langle\phi^+|) + F'|\phi^+\rangle\langle\phi^+|, \tag{92}$$

where $|\phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$, $F' = \mathrm{Tr}(\rho_{\text{iso}}|\phi^+\rangle\langle\phi^+|)$ is also the fidelity, different from the fidelity in the RGME.

The expression of the GME is given in [19]:

$$E_{\sin^2} = 1 - \frac{1}{d}(\sqrt{F'} + \sqrt{(1-F')(d-1)})^2. \tag{93}$$

We can testify when $d = 2$, $F' = \frac{1+3\alpha}{4}$, we get the GME which is equal to the RGME.

$$E_{\sin^2} = 1 - \left(\frac{1}{2}\sqrt{\frac{1+3\alpha}{2}} + \frac{1}{2}\sqrt{\frac{3(1-\alpha)}{2}}\right)^2 = \widetilde{E}_{\sin^2}, \tag{94}$$

when $d = 3$, $F' = \frac{1+8\alpha}{9}$, the GME is equal to the RGME:

$$E_{\sin^2} = 1 - \left(\frac{1}{3}\sqrt{\frac{1+8\alpha}{3}} + \frac{1}{3}\sqrt{\frac{16(1-\alpha)}{3}}\right)^2 = \widetilde{E}_{\sin^2}, \tag{95}$$

when $d = 4$, $F' = \frac{1+15\alpha}{16}$, the GME is equal to the RGME:

$$E_{\sin^2} = 1 - \left(\frac{1}{4}\sqrt{\frac{45(1-\alpha)}{4}} + \frac{1}{4}\sqrt{\frac{1+15\alpha}{4}}\right)^2 = \widetilde{E}_{\sin^2}, \tag{96}$$

when the dimension is d, $F' = \frac{1+(d^2-1)\alpha}{d^2}$, GME and RGME are also equal:

$$E_{\sin^2} = 1 - \left(\frac{d^2-1}{d}\sqrt{\frac{(1-\alpha)}{d(d+1)}} + \frac{1}{d}\sqrt{\frac{1+(d^2-1)\alpha}{d}}\right)^2 = \widetilde{E}_{\sin^2}. \tag{97}$$

These not only show that the GME and the RGME are equal for the isotropic states, but also show that our revision for the GME is reasonable. The contour figure of the isotropic state is given in figure 7. The RGME in the undertone field is greater than that in the dark field.

The concrete expression of RE [47] is

$$E_{\text{re}} = \log d + F'\log F' + (1-F')\log\frac{1-F'}{d-1}, \tag{98}$$

where $F' = \frac{1+(d^2-1)\alpha}{d^2}$.

For this class of states, we still obtain the same relation

$$\widetilde{E}_{\sin^2}(\rho) \leqslant E_{\text{re}}(\rho) \tag{99}$$

like the special states in $2 \otimes n$ quantum system by drawing figures. This result emphasizes the rationality of the revision and it is an important conclusion through this paper simultaneously.

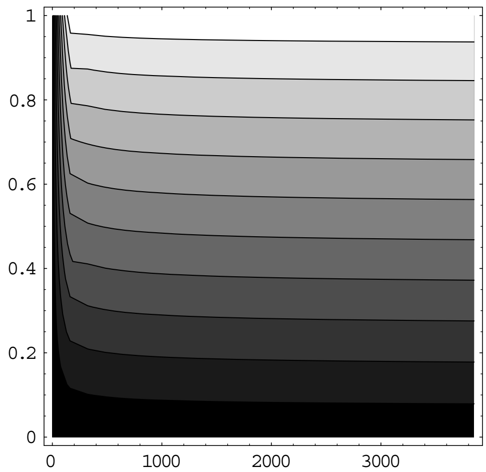

Figure 7. The contour figure for the revised geometric measure of entanglement (RGME) of the isotropic states, where undertone field denotes higher density than counterpart in the dark field.

Other measures of entanglement about the isotropic state have been obtained, concrete results are summarized in the following contexts. First, concurrence and I-concurrence also have been given in [47, 48]. The concurrence for the qubit isotropic state is

$$C = \begin{cases} 0 & F' < \frac{1}{2}, \\ 2F' - 1 & \frac{1}{2} \leqslant F' \leqslant 1, \end{cases} \tag{100}$$

qutrit isotropic state, I-concurrence (generalized concurrence) is

$$C = \begin{cases} 0 & F' < \frac{1}{3}, \\ \sqrt{3}\left(F' - \frac{1}{3}\right) & \frac{1}{3} \leqslant F' \leqslant 1. \end{cases} \tag{101}$$

In turn, qudit isotropic state, I-concurrence is

$$C = \begin{cases} 0 & F' < \frac{1}{d}, \\ \sqrt{\frac{2d}{d-1}}\left(F' - \frac{1}{d}\right) & \frac{1}{d} \leqslant F' \leqslant 1. \end{cases} \tag{102}$$

Then, the EOF for the isotropic state is

$$E_f = \begin{cases} 0, & F' < \frac{1}{d}, \\ h(x) + (1-x)\log(d-1), & \frac{1}{d} \leqslant F' < \frac{(d-1)(1-F')}{F'}, \\ \mathrm{d}\log\frac{(d-1)(F'-1)}{d-2} + \log d, & \frac{4(d-1)}{d^2} \leqslant F' \leqslant 1, \end{cases} \tag{103}$$

where $h(x) = -x\log x - (1-x)\log(1-x)$ and $x = \frac{F'}{d}\left(1 + \sqrt{\frac{(d-1)(1-F')}{F'}}\right)^2$. Note when $d \to \propto$, we have $E_f \to F'\log d$.

As for the relation about the concurrence, the RGME and the RE, we find their relation is uncertain by numerical analysis.

Now, we consider the generalized case, n-particle and d-level isotropic state [12] is expressed as follows:

$$\rho(\alpha) = (1-\alpha)\frac{I}{d^n} + \alpha|\psi^+\rangle\langle\psi^+|, \qquad 0 \leqslant \alpha \leqslant 1, \tag{104}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\cdots i\rangle$. The closest separable state is

$$\rho(\alpha_0) = (1-\alpha_0)\frac{I}{d^n} + \alpha_0|\psi^+\rangle\langle\psi^+|, \qquad \alpha_0 = \frac{1}{1+d^{n-1}}, \tag{105}$$

when $n = 3$, $d = 2$

$$F = 7\sqrt{\frac{1-\alpha}{80}} + \sqrt{\frac{3(1+7\alpha)}{80}}, \tag{106}$$

when $n = 3$, $d = 3$

$$F = 26\sqrt{\frac{1-\alpha}{810}} + \sqrt{\frac{2(1+26\alpha)}{405}}, \tag{107}$$

when $n = 4$, $d = 2$

$$F = 15\sqrt{\frac{1-\alpha}{288}} + \sqrt{\frac{1+15\alpha}{96}}. \tag{108}$$

By the mathematical induction method, we get

$$F = (d^n-1)\sqrt{\frac{1-\alpha}{d^n(d^n+d)}} + \sqrt{\frac{1+(d^n-1)\alpha}{d^n}\frac{d+1}{d^n+d}}, \tag{109}$$

notice when $n = 2$, we get formula (90) of the two-particle isotropic state again.

Finally, the explicit expression of the RGME for the generalized case is

$$\widetilde{E}_{\sin^2} = 1 - \left((d^n-1)\sqrt{\frac{1-\alpha}{d^n(d^n+d)}} + \sqrt{\frac{1+(d^n-1)\alpha}{d^n}\frac{d+1}{d^n+d}}\right)^2, \tag{110}$$

when $n = 2$, we get formula (91) of the two-particle isotropic state again. These indicate that our revision is reasonable too.

4.4. RGME of some multi-particle bound entangled states

Multi-particle entanglement exhibits a much richer structure than biparticle entanglement; even in the simplest case, the quantification of multi-particle entanglement is a hard computable problem. It is thus worth seeking cases in which one can explicitly obtain an expression to measure the amount of entanglement.

Bound multi-particle entangled states, the peculiar class of states, play an important role in many calculations of entanglement measure. Here, we analytically determine the entanglement in terms of RGME for two multi-particle bound entangled states in [20] by a purification procedure. The result shows that the RGME is equal to the GME which elucidates that the RGME is an appropriate measure of entanglement compare to other measures of entanglement, again. In order to explain explicitly, we give a requisite theorem about purification [30].

Uhlmann theorem. Assume ρ, σ are states of quantum system Q, introduce the second quantum system R with dimension greater than or equal to the dimension of Q, then

$$F(\rho, \sigma) = \max|\langle\varphi|\psi\rangle|, \tag{111}$$

where the maximum runs over all purification $|\psi\rangle$ of ρ, $|\varphi\rangle$ of σ in RQ.

First, we consider the Smolin's four-party unlockable bound entangled state:

$$\rho^{ABCD} = \frac{1}{4}\sum_{i=0}^{3}(|\psi_i\rangle\langle\psi_i|)_{AB}\otimes(|\psi_i\rangle\langle\psi_i|)_{CD}$$
$$= \frac{1}{4}\sum_{i=0}^{3}|X_i\rangle\langle X_i|, \tag{112}$$

where

$$|X_0\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \qquad |X_1\rangle = \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle),$$
$$|X_2\rangle = \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle), \qquad |X_3\rangle = \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle). \tag{113}$$

We can see ρ^{ABCD} has been written in the eigenvalue decomposition itself, that is

$$\rho^{ABCD} = \frac{1}{4}\sum_{i=0}^{3}|X_i\rangle\langle X_i| = \sum_{i=0}^{3}p_i|X_i\rangle\langle X_i|. \tag{114}$$

Assume that the closest separable state is σ, its eigenvalue decomposition is $\sigma = \sum_{i=0}^{3} q_i|\phi\rangle\langle\phi|$, where $q_0 = 1, q_1 = q_2 = q_3 = 0, |\phi\rangle = \otimes_{i=0}^{3}(c_i|0\rangle + s_i|1\rangle)$, where $c_i \equiv \cos\theta_i, s_i \equiv \sin\theta_i$ with $0 \leqslant \theta_i \leqslant \frac{\pi}{2}$, so

$$|\psi\rangle = \sum_{i=0}^{3}\sqrt{p_i}|X_i\rangle|\mathrm{i}^{R_A}\rangle, \qquad |\varphi\rangle = \sum_{i=0}^{3}\sqrt{q_i}|\phi\rangle|\mathrm{i}^{R_B}\rangle = |\phi\rangle|\mathrm{i}^{R_B}\rangle. \tag{115}$$

Because of arbitrariness of purification, we choose $\mathrm{i}^{R_A} = \mathrm{i}^{R_B}$, then

$$F = \langle\varphi|\psi\rangle = \langle\phi|\sum_{i=0}^{3}\sqrt{p_i}|X_i\rangle = \sum_{i=0}^{3}\sqrt{p_i}\langle\phi|X_i\rangle$$
$$= \sqrt{\frac{p_0}{2}}(c_1c_2c_3c_4 + s_1s_2s_3s_4) + \sqrt{\frac{p_1}{2}}(c_1c_2s_3s_4 + s_1s_2c_3c_4)$$
$$+ \sqrt{\frac{p_2}{2}}(c_1s_2c_3s_4 + s_1c_2s_3c_4) + \sqrt{\frac{p_3}{2}}(c_1s_2s_3c_4 + s_1c_2c_3s_4), \tag{116}$$

by Cauchy–Schwarz inequality $\langle v|v\rangle\langle w|w\rangle \geqslant |\langle v|w\rangle|^2$, we obtain

$$F = \sqrt{\frac{1}{2}}, \qquad \widetilde{E}_{\sin^2} = \frac{1}{2}. \tag{117}$$

In addition, [20] conjectures that its closest separable mixed state is

$$\sigma = \tfrac{1}{8}(|0000\rangle\langle 0000| + |1111\rangle\langle 1111|) + |0011\rangle\langle 0011| + |1100\rangle\langle 1100|$$
$$+ |0101\rangle\langle 0101| + |1010\rangle\langle 1010| + |0110\rangle\langle 0110| + |1001\rangle\langle 1001|. \tag{118}$$

We compute the RGME using above suspected closet separable state. The results are indeed $F = \frac{1}{\sqrt{2}}, \widetilde{E}_{\sin^2} = \frac{1}{2}$ which are same as equation (117). Hence, we show that the conjecture is valid from the inverted angle.

Next, we consider the Dur's N-party Bell-inequality-violating bound entangled states $(N \geqslant 4)$ [49]

$$\rho_N(x) = x|\psi_G\rangle\langle\psi_G| + \frac{1-x}{2N}\sum_{k=1}^{N}(P_k + \bar{P}_k), \tag{119}$$

where

$$|\psi_G\rangle = \frac{1}{\sqrt{2}}(|0^{\otimes N}\rangle + |1^{\otimes N}\rangle),$$
$$P_K = |u_k\rangle\langle u_k|, \qquad |u_k\rangle = |0\rangle_1|0\rangle_2\cdots|1\rangle_k\cdots|0\rangle_N, \tag{120}$$
$$\bar{P}_K = |v_k\rangle\langle v_k|, \qquad |v_k\rangle = |1\rangle_1|1\rangle_2\cdots|0\rangle_k\cdots|1\rangle_N.$$

The eigenvalue decomposition can be conveniently written as

$$\rho_N(x) = \sum_{i=0}^{N-1} p_i|\xi_i\rangle\langle\xi_i| = |\psi(x,\{q,r\})\rangle\langle\psi(x,\{q,r\})|,$$
$$\sigma = |\phi\rangle\langle\phi|, \tag{121}$$

where

$$|\psi(x,\{q,r\})\rangle = \sqrt{x}|\psi_G\rangle + \sqrt{1-x}\sum_{k=1}^{N}(\sqrt{q_k}|u_k\rangle + \sqrt{r_k}|v_k\rangle). \tag{122}$$

Through the way of purification, we obtain

$$|\psi\rangle = \sum_{i=0}^{N-1}\sqrt{p_i}|\xi_i\rangle|\mathrm{i}^{R_A}\rangle, \qquad |\varphi\rangle = \sum_{i=0}^{N-1}\sqrt{q_i}|\phi\rangle|\mathrm{i}^{R_B}\rangle. \tag{123}$$

Choose $\mathrm{i}^{R_A} = \mathrm{i}^{R_B}$, then we have

$$F = \langle\varphi|\psi\rangle = \langle\varphi|\sum_{i=0}^{N-1}\sqrt{p_i}|\xi_i\rangle$$
$$= \sqrt{\frac{x}{2}}(c_1\cdots c_N + s_1\cdots s_N) + \sqrt{1-x}\sum_{k=1}^{N}(\sqrt{q_k}c_1\cdots s_k\cdots c_N + \sqrt{r_k}s_1\cdots c_k\cdots s_N), \tag{124}$$

similarly, using the Cauchy–Schwarz inequality, we obtain

$$F = \sqrt{\frac{2-x}{2}}, \qquad \widetilde{E}_{\sin^2} = \frac{x}{2}. \tag{125}$$

It is clear that the GME and the RGME are equal for above two bound entangled states. A conjecture concerning the closest separable state of Dur's bound entangled state is presented in [20]. The suspected form is

$$\rho_N(x) = x(|0\cdots0\rangle\langle0\cdots0| + |1\cdots1\rangle\langle1\cdots1|) + \frac{1-x}{2N}\sum_{k=1}^{N}(P_k + \bar{P}_k). \tag{126}$$

By Uhlmann theorem, we can easily obtain

$$|\psi\rangle = \sqrt{x}|0\cdots0\rangle|\mathrm{i}^R\rangle + \sum_{k=1}^{N}\sqrt{\frac{1-x}{2N}}P_k|\mathrm{i}^R\rangle + \sum_{k=1}^{N}\sqrt{\frac{1-x}{2N}}\bar{P}_k|\mathrm{i}^R\rangle + \sqrt{x}|1\cdots1\rangle|\mathrm{i}^R\rangle,$$
$$|\varphi\rangle = \sqrt{\frac{x}{2}}|0\cdots0\rangle|\mathrm{i}^R\rangle + \sum_{k=1}^{N}\sqrt{\frac{1-x}{2N}}P_k|\mathrm{i}^R\rangle + \sum_{k=1}^{N}\sqrt{\frac{1-x}{2N}}\bar{P}_k|\mathrm{i}^R\rangle + \sqrt{\frac{x}{2}}|1\cdots1\rangle|\mathrm{i}^R\rangle. \tag{127}$$

If the overlap satisfies $\langle\varphi|\psi\rangle = \sqrt{1-\frac{x}{2}}$, then the suspected separable state is valid. But the result of computation does not satisfy this condition, so we say this conjecture is invalid.

Thus, we have presented analytical results on how much entanglement is bound in two distinct multipartite bound entanglement states using the revised measure. For these states, the RE is still an upper bound on the RGME. For example, the RE of the Smolin state is 1 [20] which is larger than its RGME $\frac{1}{2}$.

5. Conclusion

The merit of this revised measure RGME lies on its suitability for any-partite system with any dimension. The revision of the GME becomes more accurate. Because the RGME abandons the condition that the closest separable state is pure state, even for the case of the pure state, simultaneously uses the fidelity to substitute the overlap in view of the relation between the fidelity and the overlap, hence it can be expressed congruously. The essence of the problem is attributed to find out the closest separable state, naturally, we need not use the convex hull construction to consider the case of the mixed state. We have presented analytical results about measure of entanglement of some special multi-particle cases for which other measures of entanglement are bigger than RGME, hence the advantage of RGME is exhibited clearly.

Some properties of RGME are presented in the proposition form. We discover that the RGME is smaller than or equal to the EOF (or the ER) in the bipartite pure state setting. For any pure state, the RGME is smaller than or equal to the trace distance. Besides, we obtain a relation between the von Neumann entropy and the trace distance.

The revised entanglement quantifier is used to quantify the entanglement of some special states. We acquire two main bound conditions, one is $\widetilde{E}_{\sin^2}(\rho) \leqslant E_{\sin^2}(\rho)$ for the case of pure state, another is $\widetilde{E}_{\sin^2} \leqslant E_{\text{re}} \leqslant E_f \leqslant$ negativity for the two-parameter class of states in $2 \otimes n$ quantum system. The bound condition $\widetilde{E}_{\sin^2} \leqslant E_{\text{re}}$ is still valid for the bipartite maximally entangled state, isotropic state, Smolin and Dur multipartite bound states. From these conclusions, we see that our RGME is reasonable and has explicit application.

However, we should point out the disadvantage of RGME. Like the RE, the search of the closest separable state is necessary for calculation. In fact, this is a tough task. Certainly, GME and EOF use the convex hull construction to deal with the case of mixed state which is not easier than the former. Fortunately, for some special states, the closest separable state under RE is also that under RGME, which simplifies the difficulty greatly and makes the calculation realizable. From this sense, we think this quantifier outbalances other candidates of entanglement.

In order to alleviate and overcome the difficulty of finding the closest separable state, people provide many methods for calculation of RE in the literature, the convex programming method [35], the numerical value analysis method [13], etc, but they are effective only for the special scenarios. The avail method which suits to any state is to have a guess to what the minimum for a pure state should be, then use the formal proof to testify, i.e. considering the gradient, see [13].

Note an entanglement monotone derived from Grover's algorithm called the Groverian entanglement is presented in [50, 51]. For a pure state, the Groverian entanglement is equal to the GME, but its physical meanings and springboards are completely different, because the Groverian entanglement is motivated by a quantum algorithm, while the GME is motivated from a geometric viewpoint. Groverian entanglement demonstrates how well a state performs an input to Grover's search algorithm depends critically upon the entanglement presented in that state; the more the entanglement, the less well the algorithm performs. The GME is the sine of the angle between the pure state and its closest separable state; the stronger the entanglement of state becomes, the larger the angle between them is. The Groverian entanglement is introduced just for the pure state of multiple qubits, GME is suitable for any-particle system with any dimension. On the basis of the results about pure states, the Groverian entanglement is generalized to the case of mixed states [52], but the operational explanation cannot be generalized to mixed states. In this paper the GME is revised, while the RGME still maintains the inherent advantages of the GME and has clear physical meaning [53]. It happens that the forms of generalized Groverian entanglement [52] and the RGME provided in this paper are coincident. It is worth emphasizing that our work finished independently and in a different way.

To the best of our knowledge, corresponding results have not been obtained for other measures of entanglement. Recently, a connection is identified between the GME and the entanglement witnesses [21], which can in principle be measured locally. So, we can render the GME experimentally verifiable. The connection between the generalized robustness and the geometric measure of entanglement is also presented in [54]. In view of these works, we wish to find out the deeper relation between the RGME and other measures of entanglement.

In conclusion, we believe that our analysis is helpful for better understanding the essence of amount of the entanglement. Because many entangled quantifiers exist, it is important to explore their relations. We believe this should be a major goal in the theory of entanglement and hope that the discussion in this paper can give some help in this sense.

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Appendix A

In this section, we use the convex programming method to find the closest separable state of the two-parameter $2 \otimes 3$ quantum system:

$$\tilde{\rho} = \beta|\psi^+\rangle\langle\psi^+| + \gamma|\psi^-\rangle\langle\psi^-| + \beta(|00\rangle\langle 00| + |11\rangle\langle 11|) + \alpha(|02\rangle\langle 02| + |12\rangle\langle 12|). \tag{A.1}$$

In light of the method in [35], the closest separable state of $\tilde{\rho}$ has the following form:

$$\tilde{\sigma}^* = p_1|\psi^+\rangle\langle\psi^+| + p_2|\psi^-\rangle\langle\psi^-| + p_3|00\rangle\langle 00| + p_4|11\rangle\langle 11| + p_5|02\rangle\langle 02| + p_6|12\rangle\langle 12|, \tag{A.2}$$

where $\sum_i p_i = 1$. Then, the partial transpose of $\tilde{\sigma}^*$ is

$$\begin{aligned}(\tilde{\sigma}^*)^{T_B} = {} & \frac{p_1}{2}(|01\rangle\langle 01| + |00\rangle\langle 11| + |11\rangle\langle 00| + |10\rangle\langle 10|) \\ & + \frac{p_2}{2}(|01\rangle\langle 01| - |00\rangle\langle 11| - |11\rangle\langle 00| + |10\rangle\langle 10|) \\ & + p_3|00\rangle\langle 00| + p_4|11\rangle\langle 11| + p_5|02\rangle\langle 02| + p_6|12\rangle\langle 12|.\end{aligned} \tag{A.3}$$

Because $\tilde{\sigma}^*$ is separable state, we know $(\tilde{\sigma}^*)^{T_B}$ should be positive-definite or semi-positive definite matrix according to the separable criterion, so we obtain an inequality condition

$$p_3 p_4 - \left(\frac{p_1 - p_2}{2}\right)^2 \geqslant 0. \tag{A.4}$$

The RE of $\tilde{\rho}$ can be expressed as

$$\begin{aligned} S(\tilde{\rho}\|\tilde{\sigma}^*) &= \mathrm{tr}(\tilde{\rho}\log\tilde{\rho}) - \mathrm{tr}(\tilde{\rho}\|\tilde{\sigma}^*) \\ &= \mathrm{tr}(\tilde{\rho}\log\tilde{\rho}) + f(p_i). \end{aligned} \tag{A.5}$$

Let

$$F(p_i) = f(p_i) + \lambda\left(\sum_i p_i - 1\right) + \eta\left(p_3 p_4 - \left(\frac{p_1 - p_2}{2}\right)^2\right)$$
$$= -(\beta\log p_1 + \gamma\log p_2 + \beta\log p_3 + \beta\log p_4 + \alpha\log p_5 + \alpha\log p_6)$$
$$+ \lambda\left(\sum_i p_i - 1\right) + \eta\left(p_3 p_4 - \left(\frac{p_1 - p_2}{2}\right)^2\right), \tag{A.6}$$

where λ, η are Lagrange multipliers. The problem comes to solve the extremum of $f(p_i)$ with a constraint, i.e. solve the following equations set:

$$\begin{cases} \dfrac{\partial F(p_i)}{\partial p_j} = 0, & j = 1, 2, 3, 4, \\ \sum_i p_i - 1 = 0, & \eta\left(p_3 p_4 - \left(\dfrac{p_1 - p_2}{2}\right)^2\right) = 0, \quad \eta \geqslant 0. \end{cases} \tag{A.7}$$

Through calculation we get two groups of solution. The first group of solution is

$$p_1 = p_3 = p_4 = \frac{3\beta + \gamma}{6}, \qquad p_2 = \frac{3\beta + \gamma}{2}, \qquad p_5 = p_6 = \alpha. \tag{A.8}$$

The second group of solution is

$$p_1' = \frac{3\beta + \gamma}{2}, \qquad p_2' = \frac{\gamma(3\beta + \gamma)}{2(2\beta + \gamma)},$$
$$p_3' = p_4' = \frac{\beta(3\beta + \gamma)}{2(2\beta + \gamma)}, \qquad p_5' = p_6' = \alpha. \tag{A.9}$$

Due to the equation

$$f(p_i) = -(\beta\log p_1 + \gamma\log p_2 + \beta\log p_3 + \beta\log p_4 + \alpha\log p_5 + \alpha\log p_6), \tag{A.10}$$

and the relations

$$1/2 \leqslant \gamma \leqslant 1, \qquad \beta > 0, \tag{A.11}$$

we know

$$f(p_i) \leqslant f(p_i'). \tag{A.12}$$

Evidently, we choose the first group of solution. At the end, we obtain the closest separable state of $\tilde{\rho}$:

$$\tilde{\sigma}^* = \alpha(|02\rangle\langle 02| + |12\rangle\langle 12|) + \frac{3\beta + \gamma}{2}|\psi^-\rangle\langle\psi^-|$$
$$+ \frac{3\beta + \gamma}{6}(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|). \tag{A.13}$$

Appendix B

In this appendix, we show a proof that the closest separable state of two-parameter class of stats in $2 \otimes n$ quantum system

$$\rho = \alpha\sum_{i=0}^{1}\sum_{j=2}^{n-1}|ij\rangle\langle ij| + \beta(|00\rangle\langle 00| + |11\rangle\langle 11|) + \frac{\beta + \gamma}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|)$$
$$+ \frac{\beta - \gamma}{2}(|01\rangle\langle 10| + \langle 10|\langle 01|) \tag{B.1}$$

is the state

$$\sigma^* = \alpha \sum_{i=0}^{1} \sum_{j=2}^{n-1} |ij\rangle\langle ij| + \frac{3\beta+\gamma}{6}(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|) + \frac{3\beta+\gamma}{2}|\psi^-\rangle\langle\psi^-|, \tag{B.2}$$

where $|ij\rangle : i = 0, 1,\ j = 0, 1, \ldots, n-1$.

Our proof goes as follows: if σ^* is the closest separable state of entangled state ρ, then the value of differential coefficient $\frac{\mathrm{d}}{\mathrm{d}x}S(\rho\|(1-x)\sigma^* + x\sigma)$ is non-negative, where σ is any separable state. However, if σ^* was not a minimum the above gradient would be strictly negative which is a contradiction.

Proof. For any given positive operator A, we have

$$\log A = \int_0^\infty \frac{At-1}{A+t}\frac{\mathrm{d}t}{1+t^2}. \tag{B.3}$$

Let

$$f(x,\sigma) = S(\rho\|(1-x)\sigma^* + x\sigma), \tag{B.4}$$

then

$$\begin{aligned}\frac{\partial f}{\partial x}(0,\sigma) &= -\lim_{x\to 0}\mathrm{tr}\left\{\frac{\rho(\log((1-\mathrm{x})\sigma^* + \mathrm{x}\sigma) - \log\sigma^*)}{\mathrm{x}}\right\}\\ &= \mathrm{tr}\left(\rho\int_0^\infty (\sigma^*+\mathrm{t})^{-1}(\sigma^*-\sigma)(\sigma^*+\mathrm{t})^{-1}\,\mathrm{dt}\right)\\ &= 1 - \int_0^\infty \mathrm{tr}(\rho(\sigma^*+t)^{-1}\sigma(\sigma^*+t)^{-1})\,\mathrm{d}t\\ &= 1 - \int_0^\infty \mathrm{tr}((\sigma^*+t)^{-1}\rho(\sigma^*+t)^{-1}\sigma)\,\mathrm{d}t.\end{aligned} \tag{B.5}$$

By choosing the appropriate basis sequence, σ^* and ρ can be expressed in the following matrix form:

$$\sigma^* = \begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}, \qquad \rho = \begin{pmatrix} B_1 & 0\\ 0 & B_2\end{pmatrix}, \tag{B.6}$$

where

$$A_1 = \begin{pmatrix} \frac{3\beta+\gamma}{6} & 0 & 0 & 0\\ 0 & \frac{3\beta+\gamma}{3} & -\frac{3\beta+\gamma}{6} & 0\\ 0 & -\frac{3\beta+\gamma}{6} & \frac{3\beta+\gamma}{3} & 0\\ 0 & 0 & 0 & \frac{3\beta+\gamma}{6}\end{pmatrix},$$

$$B_1 = \begin{pmatrix} \beta & 0 & 0 & 0\\ 0 & \frac{\beta+\gamma}{2} & \frac{\beta-\gamma}{2} & 0\\ 0 & \frac{\beta-\gamma}{2} & \frac{\beta+\gamma}{2} & 0\\ 0 & 0 & 0 & \beta\end{pmatrix}, \tag{B.7}$$

$$A_2 = \begin{pmatrix} \alpha & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \alpha\end{pmatrix}, \qquad B_2 = \begin{pmatrix} \alpha & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \alpha\end{pmatrix}.$$

A_2, B_2 is $(2n-4)\times(2n-4)$ diagonal matrices with diagonal element α, then

$$\begin{aligned}(\sigma^*+t)^{-1}\rho(\sigma^*+t)^{-1} &= \begin{pmatrix}(A_1+t)^{-1} & 0\\ 0 & (A_2+t)^{-1}\end{pmatrix}\cdot\begin{pmatrix}B_1 & 0\\ 0 & B_2\end{pmatrix}\\ &\quad\cdot\begin{pmatrix}(A_1+t)^{-1} & 0\\ 0 & (A_2+t)^{-1}\end{pmatrix}\\ &= \begin{pmatrix}(A_1+t)^{-1}B_1(A_1+t)^{-1} & 0\\ 0 & (A_2+t)^{-1}B_2(A_2+t)^{-1}\end{pmatrix}.\end{aligned}\tag{B.8}$$

□

Let

$$g=\int_0^\infty(\sigma^*+t)^{-1}\rho(\sigma^*+t)^{-1}\,\mathrm{d}t,\tag{B.9}$$

through some calculations, we obtain

$$g=\begin{pmatrix}C_1 & 0\\ 0 & E\end{pmatrix},\tag{B.10}$$

where

$$C_1=\begin{pmatrix}\frac{6\beta}{3\beta+\gamma} & 0 & 0 & 0\\ 0 & 1 & \frac{6\beta}{3\beta+\gamma}-1 & 0\\ 0 & \frac{6\beta}{3\beta+\gamma}-1 & 1 & 0\\ 0 & 0 & 0 & \frac{6\beta}{3\beta+\gamma}\end{pmatrix},\tag{B.11}$$

where E is the identity matrix of $(2n-4)\times(2n-4)$. Because $2(n-2)\alpha+3\beta+\gamma=1$, $0\leqslant\alpha\leqslant 1/(2n-4)$ and $1/2\leqslant\gamma\leqslant 1$, we have

$$0\leqslant\frac{6\beta}{3\beta+\gamma}\leqslant 1.\tag{B.12}$$

Let $\sigma=|\eta\rangle\langle\eta|\otimes|\xi\rangle\langle\xi|$, where $|\eta\rangle=\sum_n a_n|n\rangle$ and $|\xi\rangle=\sum_n b_n|n\rangle$ are orthogonal normalization vectors, then

$$\begin{aligned}\frac{\partial f}{\partial x}(0,\sigma)-1 &= -\operatorname{tr}(g\sigma)\\ &= -\Bigg[\frac{6\beta}{3\beta+\gamma}(|a_0|^2|b_0|^2+|a_1|^2|b_1|^2)\\ &\quad+\left(\frac{6\beta}{3\beta+\gamma}-1\right)(a_0a_1^*b_1b_0^*+a_0^*a_1b_1^*b_0)\\ &\quad+|a_0|^2|b_1|^2+|a_1|^2|b_0|^2+\sum_{i=0}^{1}\sum_{j=2}^{n-1}|a_i|^2|b_j|^2\Bigg].\end{aligned}\tag{B.13}$$

Due to $0\leqslant\frac{6\beta}{3\beta+\gamma}\leqslant 1$, we get

$$\begin{aligned}\left|\frac{\partial f}{\partial x}(0,\sigma)-1\right| &= \frac{6\beta}{3\beta+\gamma}(|a_0|^2|b_0|^2+|a_1|^2|b_1|^2)+\frac{6\beta}{3\beta+\gamma}(a_0a_1^*b_1b_0^*+a_0^*a_1b_1^*b_0)\\ &\quad+|a_0b_1-a_1b_0|^2+\sum_{i=0}^{1}\sum_{j=2}^{n-1}|a_i|^2|b_j|^2\end{aligned}$$

$$
\begin{aligned}
&\leqslant (|a_0|^2|b_0|^2 + |a_1|^2|b_1|^2) + (a_0 a_1^* b_1 b_0^* + a_0^* a_1 b_1^* b_0) \\
&\quad + |a_0 b_1 - a_1 b_0|^2 + \sum_{i=0}^{1}\sum_{j=2}^{n-1} |a_i|^2 |b_j|^2 \\
&= |a_0|^2|b_0|^2 + |a_1|^2|b_1|^2 + |a_0|^2|b_1|^2 + |a_1|^2|b_0|^2 + \sum_{i=0}^{1}\sum_{j=2}^{n-1} |a_i|^2|b_j|^2 \\
&\leqslant 1,
\end{aligned}
\tag{B.14}
$$

i.e.,

$$
\frac{\partial f}{\partial x}(0, \sigma) \geqslant 0. \tag{B.15}
$$

For any separable state σ, which can be expressed by $\sigma = \sum_i p_i |\eta^i \xi^i\rangle\langle\eta^i \xi^i|$, then

$$
\frac{\partial f}{\partial x}(0, \sigma) = \sum_i p_i \frac{\partial f}{\partial x}(0, |\eta^i \xi^i\rangle\langle\eta^i \xi^i|) \geqslant 0, \tag{B.16}
$$

this proves that σ^* is the closest separable state of ρ for certain.

Appendix C

Here, we use mathematical induction method to prove the formula of fidelity of two-parameter class of states in $2 \otimes n$ quantum system:

$$
F_n = 2(n-2)\alpha + \frac{3\sqrt{\beta(3\beta+\gamma)}}{\sqrt{6}} + \frac{\sqrt{\gamma(3\beta+\gamma)}}{\sqrt{2}}. \tag{C.1}
$$

Proof. When $n = 3$, it is easy to obtain the fidelity of $2 \otimes 3$ quantum system

$$
F_3 = 2\alpha + \frac{3\sqrt{\beta(3\beta+\gamma)}}{\sqrt{6}} + \frac{\sqrt{\gamma(3\beta+\gamma)}}{\sqrt{2}}, \tag{C.2}
$$

when $n = 4$, the fidelity of $2 \otimes 4$ quantum system

$$
F_4 = 4\alpha + \frac{3\sqrt{\beta(3\beta+\gamma)}}{\sqrt{6}} + \frac{\sqrt{\gamma(3\beta+\gamma)}}{\sqrt{2}}. \tag{C.3}
$$

Obviously, equation (C.1) is valid for the cases of $n = 3, 4$.

Now, let us assume equation (C.1) is valid for $n = k$, i.e.,

$$
F_k = 2\,(k-2)\,\alpha + \frac{3\sqrt{\beta(3\beta+\gamma)}}{\sqrt{6}} + \frac{\sqrt{\gamma(3\beta+\gamma)}}{\sqrt{2}},
$$

then when $n = k+1$, by the definition of RGME, we know

$$
F_{k+1} = \operatorname{tr}\sqrt{\rho^{\frac{1}{2}}\sigma^*\rho^{\frac{1}{2}}}, \tag{C.4}
$$

where the matrix expressions of ρ, σ^* have been given in (B.6), (B.7).

By calculation, we obtain the fidelity

$$
F_{k+1} = \operatorname{tr}\begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix}, \tag{C.5}
$$

where matrix V is a $(2(k+1)-4)\times(2(k+1)-4)$ matrix with diagonal element α. W is a diagonal matrix which can be expressed as

$$W=\begin{pmatrix}\sqrt{\frac{\beta(3\beta+\gamma)}{6}} & 0 & 0 & 0\\ 0 & \sqrt{\frac{\beta(3\beta+\gamma)}{6}} & 0 & 0\\ 0 & 0 & \sqrt{\frac{\beta(3\beta+\gamma)}{6}} & 0\\ 0 & 0 & 0 & \sqrt{\frac{\gamma(3\beta+\gamma)}{2}}\end{pmatrix}, \tag{C.6}$$

hence

$$\begin{aligned}F_{k+1}&=2(k-1)\alpha+3\sqrt{\frac{\beta(3\beta+\gamma)}{6}}+\sqrt{\frac{\gamma(3\beta+\gamma)}{2}}+(2(k+1)-2k)\alpha\\&=2((k+1)-2)\alpha+3\sqrt{\frac{\beta(3\beta+\gamma)}{6}}+\sqrt{\frac{\gamma(3\beta+\gamma)}{2}}.\end{aligned} \tag{C.7}$$

That is, when $n=k+1$, equation (C1) is also valid. Hence, the proof is over. □

Appendix D

Here, we prove that the formula of fidelity of maximally entangled mixed states in $2\otimes n$ quantum system is

$$F=\lambda_2+\lambda_3+\frac{(\lambda_1+2\lambda_4)}{2}\sqrt{\frac{\lambda_4}{\lambda_1+\lambda_4}}+\lambda_1\sqrt{\frac{\lambda_1+2\lambda_4}{2(\lambda_1+\lambda_4)}}, \tag{D.1}$$

which is independent of n.

Proof. When $n=3,4$, it is easy to obtain the fidelity of maximally entangled mixed quantum states by straightforward matrix calculation:

$$F=\lambda_2+\lambda_3+\frac{(\lambda_1+2\lambda_4)}{2}\sqrt{\frac{\lambda_4}{\lambda_1+\lambda_4}}+\lambda_1\sqrt{\frac{\lambda_1+2\lambda_4}{2(\lambda_1+\lambda_4)}}. \tag{D.2}$$

We must calculate the fidelity for different integer n, above formula is still valid.

It is known that matrix ρ' has four eigenvalues $\lambda_1,\lambda_2,\lambda_3,\lambda_4$. According to equation (66), it can be expressed as

$$\rho'=\begin{pmatrix}\lambda_4 & 0 & 0\\ 0 & P_1 & 0\\ 0 & 0 & P_2\end{pmatrix}, \tag{D.3}$$

where

$$P_1=\begin{pmatrix}\frac{\lambda_1}{2} & 0 & -\frac{\lambda_1}{2}\\ 0 & R & 0\\ -\frac{\lambda_1}{2} & 0 & \frac{\lambda_1}{2}\end{pmatrix},\qquad P_2=\begin{pmatrix}0 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \lambda_2\end{pmatrix}. \tag{D.4}$$

P_2 is a diagonal $(n-1)\otimes(n-1)$ matrix whose last element of diagonal line is λ_2. Like matrix P_2, R is $(n-2)\otimes(n-2)$ matrix, but the last element of diagonal line is λ_3.

While the matrix expression of σ' can be written in the form

$$\sigma' = \begin{pmatrix} \frac{(\lambda_1+2\lambda_4)^2}{4(\lambda_1+\lambda_4)} & 0 & 0 \\ 0 & Q_1 & 0 \\ 0 & 0 & Q_2 \end{pmatrix}, \tag{D.5}$$

where

$$Q_1 = \begin{pmatrix} \frac{\lambda_1(\lambda_1+2\lambda_4)}{4(\lambda_1+\lambda_4)} & 0 & -\frac{\lambda_1(\lambda_1+2\lambda_4)}{4(\lambda_1+\lambda_4)} \\ 0 & R & 0 \\ -\frac{\lambda_1(\lambda_1+2\lambda_4)}{4(\lambda_1+\lambda_4)} & 0 & \frac{\lambda_1(\lambda_1+2\lambda_4)}{4(\lambda_1+\lambda_4)} \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} \frac{\lambda_1^2}{4(\lambda_1+\lambda_4)} & 0 \\ 0 & P_2' \end{pmatrix}. \tag{D.6}$$

Similarly to matrix P_2, P_2' is just $(n-2)\otimes(n-2)$ matrix.

Now ρ', σ' are all expressed in diagonal form, then we can write

$$F = \mathrm{tr}\sqrt{\rho'^{\frac{1}{2}}\sigma'\rho'^{\frac{1}{2}}} = \mathrm{tr}\begin{pmatrix} \frac{\lambda_1+2\lambda_4}{2}\sqrt{\frac{\lambda_4}{\lambda_1+\lambda_4}} & 0 & 0 \\ 0 & \sqrt{P_1^{\frac{1}{2}}Q_1P_1^{\frac{1}{2}}} & 0 \\ 0 & 0 & \sqrt{P_2^{\frac{1}{2}}Q_2P_2^{\frac{1}{2}}} \end{pmatrix}. \tag{D.7}$$

The fidelity is simplified to

$$F = \frac{(\lambda_1+2\lambda_4)}{2}\sqrt{\frac{\lambda_4}{\lambda_1+\lambda_4}} + \mathrm{tr}\left(\sqrt{P_1^{\frac{1}{2}}Q_1P_1^{\frac{1}{2}}}\right) + \mathrm{tr}\left(\sqrt{P_2^{\frac{1}{2}}Q_2P_2^{\frac{1}{2}}}\right). \tag{D.8}$$

Here, we use the fact that if a matrix has the form

$$\begin{pmatrix} x & 0 & 0 & -x \\ 0 & O & 0 & 0 \\ 0 & 0 & y & 0 \\ -x & 0 & 0 & x \end{pmatrix}, \tag{D.9}$$

where O denotes block matrix with all elements 0, then the eigenvalues of the square root of this matrix are $\sqrt{2x}$, $\sqrt{y}$.

In order to get the matrix trace, we start to calculate the eigenvalues for simplification. Note matrix $P_1^{\frac{1}{2}}Q_1P_1^{\frac{1}{2}}$ can be expressed in the form (D.10). No matter what the value of n is, matrix $\sqrt{P_1^{\frac{1}{2}}Q_1P_1^{\frac{1}{2}}}$ just has two eigenvalues, i.e. they are λ_3, $\lambda_1\sqrt{\frac{\lambda_1+2\lambda_4}{2(\lambda_1+\lambda_4)}}$. The matrix $P_2^{\frac{1}{2}}Q_2P_2^{\frac{1}{2}}$ is diagonal matrix with last element λ_2^2 in the diagonal line.

Thus, we acquire the fidelity according to equation (D.9):

$$F = \lambda_2 + \lambda_3 + \frac{(\lambda_1+2\lambda_4)}{2}\sqrt{\frac{\lambda_4}{\lambda_1+\lambda_4}} + \lambda_1\sqrt{\frac{\lambda_1+2\lambda_4}{2(\lambda_1+\lambda_4)}}. \tag{D.10}$$

It is independent of n, that is, when n is an arbitrary integer, equation (D1) is also valid, the proof is finished. □

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